Monday Math Madness 14: Shake hands Problem

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1 Problem statement

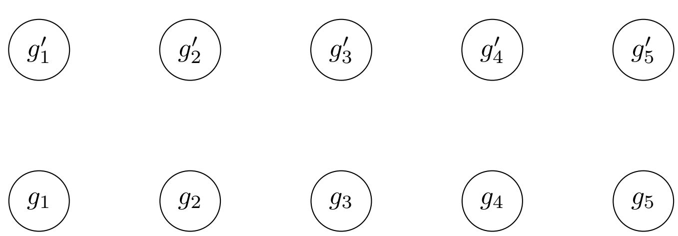
The problem statement taken from http://www.blinkdagger.com/ was the following:

> Every Friday night there is a Pizza Party at the blinkdagger mansion. On this particular Friday, Quan and Daniel have invited Rob, Jeff, and Sol to join the fun. Each person is allowed to bring one guest which brings the number of participants to 10. [...]
>
> Upon arrival, all of the guest exchange greetings, with some people shaking hands. Being the astute engineer that he is, Quan notices that:
>
> 1. No one shakes their own hand.
> 2. No one shakes the hand of the person he/she came with.
> 3. No one shakes the same person's hand more than once.
>
> After the exchange, Quan asks each person how many hands they each shook. Everyone tells him a different number.
>
> How many hands did Quan shake?

2 Thoughts and solution

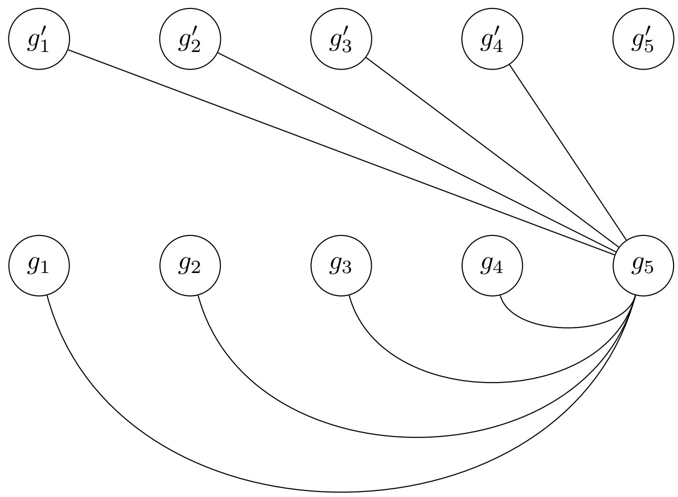
First of all, I analysed the three things Quan recognises. Because of the arguments 1 and 2, *every* guest can shake hands with maximum 8 People, because he doesn't shake hand with himself and not with the person he/she came with.

From now on, I choose a graph representation of the situation, where each node represents a guest (g_i is the i-th invited person and g_i' the person to come with g_i) and the edges are standing for handshakes. That means, there are no edges between g_i and g_i' for all $i = 1 \ldots 5$. Furthermore, there are no edges from any node to itself. Last, there is maximum one edge between to nodes. The picture below shows a situation, where no handshakes were done (because there is not one single edge between two nodes).



As we all know, in graph theory the node positions on the paper are not relevant. As explained before, each guest does maximum 8 handshakes. Of course, the minimum number of handshakes per person is 0. Because Quan asks 9 people and everybody tells him a different number, we can conclude that Quan *must* have heard 9 different numbers, that must be $0, 1, 2, \ldots, 7, 8$. That means, our final solution needs guest that shake $0, 1, 2, \ldots, 7, 8$ hands.

Now we are going to construct a solution, where all different numbers occur. As explained before, each guest can shake maximum 8 hands. Because of that, we first take a guest and "assign" him 8 handshakes. Obviously, the persons he shakes hands with are determined.

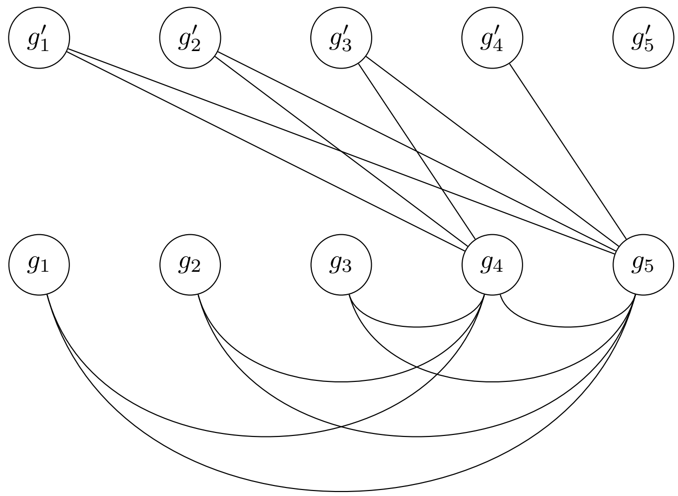


So we assigned g_5 8 handshakes, which could be done without loss of generality (because you can name the nodes as you want). So we see, that *every* node – except g'_5 has handshakes now.

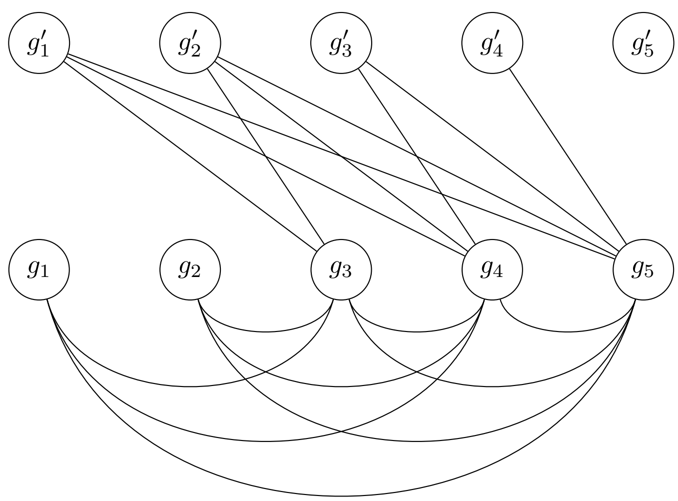
Furthermore we know now, that only one person can have 8 handshakes (here we chose g_5 to shake 8 hands). You can easily see, that it's not possible to assign 8 handshakes to any other person, so that all given conditions are fulfilled and you still have a person that has 0 handshakes. So we see, that if a person shakes 8 hands, he shakes hands with every guest except himself and the guest he came with. So if two different guests had 8 handshakes, there would be no person with 0 handshakes.

From that we conclude, that g'_5 must be the guest that has not a single handshake. From now on, we are not allowed to handshake with g_5 or g'_5 anymore, because g_5 would have 9 handshakes (which is not allowed) and g'_5 would have 1 handshake, so nobody had 0 handshakes (but indeed we need one person to have 0 handshakes).

Now we are going to assign 7 handshakes to the next person. For that, we may not choose g_5 or g'_5, es explained before. So we choose g_4 (again without loss of generality) and assign him 6 new handshakes (g_4 already has one handshake to g_5), so g_4 has 7 handshakes total. g_4 may not shake hands with g'_4 (because g_4 and g'_4 came together), g_5 (g_5 would have 9 handshakes, which is not allowed) and g'_5 (if g'_5 got a handshake we would have no person with 0 handshakes). Thus we get the following:



Now, we see, that g_4 has 7 handshakes and g'_4 has 1 handshake (to g_5), g_5 has 8 handshakes and g'_5 has 0 handshakes. All the other nodes have 2 handshakes. That means, g'_4 *must* be the guest, that has 1 and only 1 handshake. So we see, that we can't use g_4, g'_4, g_5, g'_5 for any further handshakes [1] , so we have to assign 4 handshakes to one of the remaining nodes, so we have a node that does 6 handshakes. I will choose g_3 (as always – without loss of generality), which leads us to the following situation:

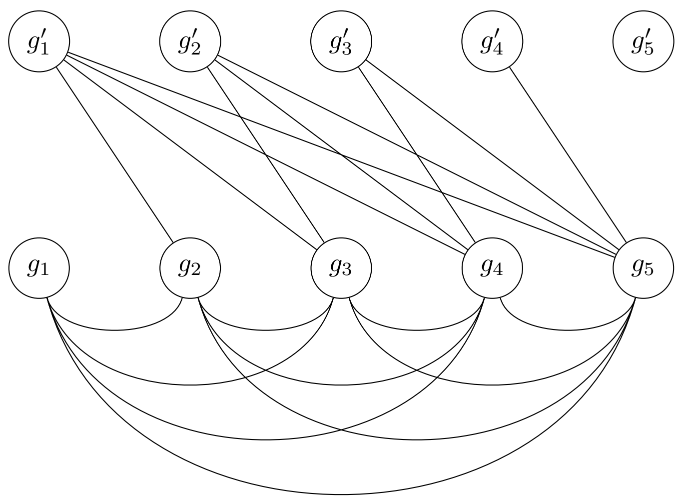


Now we have nodes and their number of handshakes:

[1]Analogous considerations as before: g_4 cannot have a further handshake, because then he would have 8 handshakes, but g_5 has already 8 handshakes. g'_4 must be the guest with 1 handshake (all the other guests have either 2 or more handshakes or need to have 0 handshakes). g'_5 cant get a handshake, because otherwise there would be no person with 0 handshakes.

Guest	Number of handshakes
g_1	3
g_1'	3
g_2	3
g_2'	3
g_3	6
g_3'	2
g_4	7
g_4'	1
g_5	8
g_5'	0

That leads us to the final assignment. Analogous thoughts as done before show, that we can only choose g_1, g_2, g_1' or g_2' for assigning 5 handshakes. Again, the assignment can be done without loss of generality. I choose g_2, so we get the following situation:



This is the final situation:

Guest	Number of handshakes
g_1	4
g_1'	4
g_2	5
g_2'	3
g_3	6
g_3'	2
g_4	7
g_4'	1
g_5	8
g_5'	0

Because Quan was told every number from 0 to 8 one time, we know, that Quan *must* be g_1 or g_1', so he did shake 4 hands.